UNIT-I **Differential equations**

Differential equations of first order and first degree

Differential equation :-

An equation involving Differentials or it's derivatives with respect to one or more independent variable and one dependent variable is called a Differential equations.

Ex:- $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^x$

Ex:- $\left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2 = 4z$

Differential equations are two types there are:-

1) Ordinary Differential equation
2) partial Differential equation.

1) Ordinary Differential equation :- A differential equation is said to be ordinary if the derivatives in the equation have reference to a single independent variable is called ordinary differential equation.

Ex:- 1) $\left(\frac{dy}{dx}\right)^3 - 4\left(\frac{dy}{dx}\right)^2 + 7y = \cos x$

2) $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = x$

2) partial differential equation:- A differential equation is said to be partial if the derivatives in the equation have reference to two (or) more independent variable is called partial differential equation

Ex:- $(y+z)\frac{dz}{dx} + (z+x)\frac{dz}{dy} = x+y$

1. Differential Equations

Bionomial Theorem :-

$(a+b)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1} b + {}^nC_2 a^{n-2} b^2 + \ldots\ldots + {}^nC_k a^{n-k} b^k + \ldots\ldots + {}^nC_n b^n$

$(a+b)^2 = 2C_0 a^2 + 2C_1 a^{2-1} b + 2C_2 b^2$

$= a^2 + 2ab + b^2$

$(a+b)^3 = 3C_0 a^3 + 3C_1 a^{3-2} b + 3C_2 a^{3-2} b + 3C_3 b^3$

$= a^3 + 3a^2b + 3ab^2 + b^3 \quad 3C_2 = \frac{3!}{(3-2)!2!} = \frac{3\times2\times1}{1!2!} = \frac{3\times2}{1\times2\times1} = 3$

$(a+b)^4 = 4C_0 a^4 + 4C_1 a^{4-1} b + 4C_2 a^{4-2} b^2 + 4C_3 a^{4-3} b^3 + 4C_4 b^4$

$= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 \quad \frac{4!}{(4-3)!3!} = \frac{4\times3\times2\times1}{1!3\times2\times1} = 4$

$(a+b)^5 = 5C_0 a^5 + 5C_1 a^{5-1} b + 5C_2 a^{5-2} b^2 + 5C_3 a^{5-3} b^3 + 5C_4 a^{5-4} b^4 + 5C_5 b^5$

$\frac{5!}{(5-1)!1!} = \frac{5\times4\times3\times2}{3\times2\times1\times2} = \frac{20}{2} = 10 \quad \frac{4!}{(4-2)!4} \quad \frac{4\times3\times2\times1}{2\times1\times2\times1} = 6$

$= a^5 + 5a^4b + 10a^3b^3 + 10a^2b^3 + 5ab^4 + b^5$

* Solve $\frac{d}{dx}(\sin x \cos x)$

$\frac{d}{dx}(\sin x \cos x) = i \sin x \frac{d}{dx}(\cos x) + \cos x \frac{d}{dx} \sin x$

$= \sin x(-\sin x) + \cos x(\cos x)$

$= -\sin^2 x + \cos^2 x$

* Solve $\int \sin x \cos x \, dx$

$\int uv\,dx = u\int v\,dx - \int\left[\frac{du}{dx}\right]\int v\,dx)\,dx$

$u = \sin x, \; v = \cos x$

$\int \sin x \cos x\,dx = \sin x \int \cos x\,dx - \int \frac{d}{dx}\sin x \int(\cos x\,dx)\,dx$

$= \sin x(\sin x) - \int \cos x(\sin x)\,dx$

$\int \sin x \cos x\,dx = \sin^2 x - \int \cos x \sin x\,dx$

$\int \sin x \cos x\,dx + \int \sin x \cos x\,dx = \sin^2 x$

$\cancel{2}\int \sin \cos x\,dx = \frac{\sin^2 x}{2}$

$I_n = \int \cos^2 x\,dx$

Solve $\int \underset{u}{x} \underset{v}{\cos x}\, dx$

$$\int uv\,dx = u\int v\,dx - \left[\int \frac{du}{dx}\int v\,dx\right]dx$$

$$\int x\cos x\,dx = x\int \cos x\,dx - \int \frac{dx}{dx}\left[\int \cos x\,dx\right]dx$$

$$= x(\sin x) - \int 1(\sin x)\,dx$$

$$= x\sin x - (-\cos x)$$

$$= x\sin x + \cos x$$

Solve $\int x^2 \sin x\, dx$

$$\int uv\,dx = u\int v\,dx - \int \frac{du}{dx}\int v\,dx]\,dx$$

$u = x^2$ $\quad v = \sin x$

$$\int x^2 \sin x\,dx = x^2\int \sin x\,dx - \left[\int \frac{d}{dx}x^2\int \sin x\,dx\right]dx$$

$$= x^2(-\cos x) - \int 2x(-\cos x)\,dx$$

$$= -x^2\cos x - 2\int x\cos x\,dx$$

$$= -x^2\cos x - 2\left[x\int \cos x\,dx - \left[\int \frac{d}{dx}x\int \cos x\,dx\right]dx\right.$$

$$= -x^2\cos x - 2\left[x(\sin x) - \int \sin x\right]dx$$

$$= -x^2\cos x - 2(x\sin x - (-\cos x))$$

$$\int x^2\sin x\,dx = -x^2\cos x - 2\left[x\sin x + \cos x\right]$$

Solve $I = \int \underset{u}{e^{ax}} \underset{v}{\cos bx}\, dx$

$$I = e^{ax}\int \cos bx\,dx - \left[\int \frac{d}{dx}e^{ax}\int \cos bx\right]dx$$

$$= e^{ax}\left[\frac{\sin bx}{b}\right] - \int a e^{ax}\left[\frac{\sin bx}{b}\right]dx$$

$$= e^{ax}\left[\frac{\sin bx}{b}\right] - \frac{a}{b}\int e^{ax}\sin bx\,dx$$

$$= e^{ax}\left[\frac{\sin bx}{b}\right] - \frac{a}{b}\left[e^{ax}\int \sin bx - \int\left[\frac{d}{dx}e^{ax}\int \sin bx\, dx\right]dx\right.$$

$$= e^{ax}\left[\frac{\sin bx}{b}\right] - \frac{a}{b}\left[e^{ax}\left(-\frac{\cos bx}{b}\right) - \int a e^{ax}\left[-\frac{\cos bx}{b}\right]dx\right.$$

$$= e^{ax}\left(\frac{\sin bx}{b}\right) - \frac{a}{b}\left[e^{ax} - \left(\frac{\cos bx}{b}\right) + \frac{a}{b}\int e^{ax}\cos bx\, dx\right.$$

$$= b\, e^{ax}\left(\frac{\sin bx}{b^2}\right) + \frac{a e^{ax}\cos bx}{b^2} - \frac{a^2}{b^2} I$$

$$I + \frac{a^2}{b^2} I = \frac{e^{ax}}{b^2}\left[b \sin bx + a\cos bx\right)$$

multiplying b^2 on both sides

$$I b^2 + \frac{a^2 \cancel{b^2}}{\cancel{b^2}} I = \frac{\cancel{b^2}\, e^{ax}(b\sin bx + a\cos bx)}{\cancel{b^2}}$$

$$Ib^2 + Ia^2 = e^{ax}(b\sin bx + a\cos bx)$$

$$I(a^2+b^2) = e^{ax}(b\sin bx + a\cos bx)$$

$$I = \frac{e^{ax}(b\sin bx + a\cos bx)}{a^2+b^2}$$

Solve $I = \int \underset{u}{e^{ax}}\, \underset{v}{\sin bx}\, dx$

$$I = e^{ax}\int \sin bx\, dx - \left[\int \frac{d}{dx}e^{ax}\int \sin bx\, dx\right] dx$$

$$= e^{ax}\left(-\frac{\cos bx}{b}\right) - \int a e^{ax}\left(-\frac{\cos bx}{b}\right) dx$$

$$= -e^{ax}\left(\frac{\cos bx}{b}\right) + \frac{a}{b}\int e^{ax}\cos bx\, dx$$

$$\cancel{= -e^{ax}\left(\frac{\cos bx}{b}\right) + \frac{a}{b}\left[e^{ax}\left(-\frac{\sin bx}{b}\right) - \int a e^{ax} \sin}$$

$$= -e^{ax}\left(\frac{\cos bx}{b}\right) + \frac{a}{b}\left[e^{ax}\int \cos bx - \int\left[\frac{d}{dx}e^{ax}\int \cos bx\, dx\right]dx\right.$$

$$= -e^{ax}\left(\frac{\cos bx}{b}\right) + \frac{a}{b}\left[e^{ax}\left(\frac{\sin bx}{b}\right) - \int a e^{ax}\left(\frac{\sin bx}{b}\right)dx\right.$$

$$= -e^{ax}\left(\frac{\cos bx}{b}\right) + \frac{a}{b}\left[e^{ax}\left(\frac{\sin bx}{b}\right) - \int a e^{ax}\left(-\frac{\sin bx}{b}\right)dx\right.$$

$$= -e^{ax}\left(\frac{\cos bx}{b}\right) + \frac{a}{b}e^{ax}\left(\frac{\sin bx}{b}\right) + \frac{a}{b}\int e^{ax}\sin bx\, dx$$

$$= -e^{ax}\left(\frac{\cos bx}{b}\right) - \frac{a e^{ax}\sin bx}{b^2} + \frac{a^2}{b^2} I$$

$$I - \frac{a^2}{b^2} I = \frac{-b e^{ax}\cos bx}{b^2} - \frac{a e^{ax}\sin bx}{b^2}$$

$$I - \frac{a^2}{b^2} I = \frac{-b e^{ax}\cos bx - a e^{ax}\sin bx}{b^2}$$

multiple b^2 on both sides.

$$b^2 I - \frac{a^2 \cancel{b^2}}{\cancel{b^2}} I = \cancel{b^2}\left[\frac{-b e^{a^2}\cos bx - a e^{ax}\sin bx}{\cancel{b^2}}\right]$$

$$I\cdot(b^2 - a^2) = -b e^{ax}\cos bx - a e^{ax}\sin bx$$

$$I = \frac{-b e^{ax}\cos bx - a e^{ax}\sin bx}{b^2 - a^2}$$

Linear differential equations :-

Definition :- A differential equation of the form $\frac{dy}{dx} + Py = Q$, where P and Q are functions of x only is called a linear differential equation of the first order in y.

$$\frac{dy}{dx} + P(x)y = Q(x)$$

Integrating factor (IF) $= e^{\int P dx}$

General solution $y(IF) = \int (IF) Q \, dx + c$

Solve $\frac{dy}{dx} + 2y = 0$

$P = 2 \qquad Q = 0$

$IF = e^{\int P dx}$

$= e^{\int 2 dx} \Rightarrow e^{2\int dx} = e^{2x}$

General solution $y(IF) = \int (IF) Q \, dx + c$

$y(e^{2x}) = \int e^{2x} . Q \, dx + c$

$y e^{2x} = c$

$y = c e^{-2x}$

Solve $\frac{dy}{dx} + 5y = 6$

$P = 5, \quad Q = 6$

$IF = e^{\int P dx}$

$= e^{\int 5 dx}$

$= e^{5x}$

General solution $y(e^{5x}) = \int e^{5x} . 6 \, dx + c$

$y(e^{5x}) = 6 \frac{e^{5x}}{5} + c$

Solve $\frac{dy}{dx} + \frac{1}{x} y = 10$

$P = \frac{1}{x} \quad Q = 10$

$IF = e^{\int P dx}$

$= e^{\int \frac{1}{x} dx}$

$= e^{\log_e x} = x$

General solution $y(IF) = \int (IF) Q \, dx + c$

$y(x) = \int x (10) \, dx + c$

$= 10 \int x \, dx + c$

$= 10 \frac{x^{1+1}}{1+1} + c$

$= \cancel{10}^{5} \frac{x^2}{\cancel{2}} + c$

$y(x) = 5x^2 + c$

$xy = 5x^2 + c$

Solve $\frac{dy}{dx} + \frac{1}{x} y = 10e^x$

$u \int v \, dx - \left[\int \frac{d}{dx} u \int v \, dx \right] dx \Big] + c$

$P = \frac{1}{x} \quad Q = \cancel{10e^x} \, 10e^x$

$IF = e^{\int \frac{1}{x} dx}$

$= e^{\log e^x}$

$IF = x$

General solution $y(IF) = \int (IF) \, Q \, dx + c$

$xy = \int x \, 10 \, e^x \, dx + c$

$xy = 10 \int \underset{u}{x} \, \underset{v}{e^x} \, dx + c$

$= 10 \left[x \int e^x dx - \left[\int \frac{d}{dx} x \int e^x dx \right] dx \right] + c$

$= 10 \left[x e^x - \left[\int (1) e^x \, dx \right] \right] + c$

$= 10 (x e^x - e^x) + c$

$xy = 10 \left[e^x (x-1) \right] + c$

- $\frac{dy}{dx} + \frac{1}{x^2} y = 10 e^{\frac{1}{x}}$

$P = \frac{1}{x^2} \quad Q = 10 e^{\frac{1}{x}}$

$IF = e^{\int \frac{1}{x^2} dx}$

$= \cancel{e^{\log e^{2x}}} = e^{\left[-\frac{1}{x}\right]}$

$= e^{-1/x}$

$\cancel{IF = 2x}$

Generesal solution $\cancel{y(2x) = \int (2x)(10 e^{\frac{1}{x}} dx) + c}$

$\cancel{2xy = 2 \cdot 10 \int x e^{\frac{1}{x}} dx + c}$

$\cancel{2xy = 20 \left[x \int e^{\frac{1}{x}} dx - \left[\int \frac{d}{dx} x \int e^{\frac{1}{x}} dx \right] + c \right.}$

$\cancel{2xy = 20 \left[x \right.}$

procedure

$\frac{dx}{dy} + Px = Q$

Intigral Factor $= e^{\int P dx}$

General solution $x(IF) = \int Q (IF) dy + c$

Solve $\frac{dy}{dx}(x^2y^3 + xy) = 1$

$x^2 y^3 + xy = \frac{dx}{dy}$

$x^2 y^3 = \frac{dx}{dy} - xy$ dividing by x^2

$y^3 = \frac{1}{x^2} \frac{dx}{dy} - \frac{xy}{x^2}$

$y^3 = \frac{1}{x^2} \frac{dx}{dy} - y \frac{1}{x}$

$\frac{1}{x^2} \frac{dx}{dy} - y \frac{1}{x} = y^3$

$v = \frac{1}{x} = x^{-1}$

$\frac{dv}{dy} = -1 x^{-1-1} \frac{dx}{dy}$

$\frac{dv}{dy} = -1 x^{-2} \frac{dx}{dy}$

$\frac{du}{dy} = -1 \frac{1}{x^2} \frac{dx}{dy}$

$\frac{1}{x^2} \frac{dx}{dy} = -\frac{du}{dy}$

$-\frac{du}{dy} - yu = y^3$

dividing by '-' on both sides

$\frac{du}{dy} + yu = -y^3$

comparing with

$\frac{du}{dy} + P(x) y\, u = Q(x) y$

$P = y \qquad Q = -y^3$

$IF = e^{\int P dy}$

$= e^{\int y dy} = e^{\frac{y^{1+1}}{1+1}} = e^{\frac{y^2}{2}}$

General solution $u(IF) = \int Q(IF)\, dy + c$

$ue^{\frac{y^2}{2}} = \int -y^3 (e^{\frac{y^2}{2}})\, dy + c$

$= -\int y^3 e^{\frac{y^2}{2}}\, dy + c$

$t = \frac{y^2}{2} \Rightarrow dt = \frac{2y}{2} dy \Rightarrow dy = \frac{dt}{y}$

$= -\int y^3 e^t \frac{dt}{dy} + c$

$= -\int y^2 e^t\, dt + c$

$= -\int 2t e^t\, dt + c$

$= -2 \int t e^t\, dt + c$

$= -2 \left[t \int e^t dt - \int \frac{dt}{dt} \int e^t dt \right] dt + c$

$= -2 \left(te^t - \int 1 e^t dt + c \right.$

$= -2 \left(te^t - e^t \right) + c$

$= -2te^t + 2e^t + c$

$$ue^{\frac{y^2}{2}} = -\cancel{x}\,\frac{y^2}{\cancel{x}}\,e^{\frac{y^2}{2}} + 2e^{\frac{y^2}{2}} + c$$

$$ue^{\frac{y^2}{2}} = -y^2 e^{\frac{y^2}{2}} + 2e^{\frac{y^2}{2}} + c$$

$$\frac{1}{x} e^{\frac{y^2}{2}} = -y^2 e^{\frac{y^2}{2}} + 2e^{\frac{y^2}{2}} + c$$

Solve $\frac{dy}{dx} + 2xy = e^{-x^2}$

$P = 2x \qquad Q = e^{-x^2}$

$$IF = e^{\int P\,dx}$$
$$= e^{\int 2x\,dx}$$
$$= e^{2\int x\,dx}$$
$$= e^{\cancel{2}\frac{x^2}{\cancel{2}}}$$
$$IF = e^{x^2}$$

$$y(IF) = \int Q\,(IF)\,dx + c$$

$$ye^{x^2} = \int e^{-x^2} e^{x^2}\,dx + c$$

$$ye^{x^2} = \int 1\,dx + c$$

$$ye^{x^2} = x + c$$

Solve $\cos^2 x \frac{dy}{dx} + y = \tan x$

$$\frac{dy}{dx} + \frac{1}{\cos^2 x} y = \frac{\tan x}{\cos^2 x}$$

$$\frac{dy}{dx} + \sec^2 x\, y = \tan x \sec^2 x$$

$P = \sec^2 x \qquad Q = \tan x \sec^2 x$

$$IF = e^{\int P\,dx}$$
$$= e^{\int \sec^2 x\,dx}$$
$$= e^{\tan x}$$

General solution $= y(IF) = \int Q(IF)\,dx + c$

$$ye^{\tan x} = \int \tan x \sec^2 x\,(e^{\tan x})\,dx + c$$

$t = \tan x$

$dt = \sec^2 x\, dx$

$dx = \frac{dt}{\sec^2 x}$

$= \int t\, e^t\, dt + c$

$= \left[t \int e^t dt - \int \frac{dt}{dt} \int e^t dt\right] dt + c$

$= t e^t - \int e^t dt + c$

$= t e^t - e^t + c$

$y e^{\tan x} = \tan x\, e^{\tan x} - e^{\tan x} + c$

$\frac{dy}{dx} + P(x) y = Q(x)$

$IF = e^{\int P(x)\, dx}$

$y(IF) = \int Q(x)(IF)\, dx + c$

$\frac{dx}{dy} + P(y) x = Q(y)$

$IF = e^{\int P(y)\, dy}$

G. S $x(I.F) = \int Q(y)(IF)\, dy + c$

Solve $(x + 2y^3)\frac{dy}{dx} = y$

$x + 2y^3 = \frac{y}{\frac{dy}{dx}}$

$x + 2y^3 = y \frac{dx}{dy}$

$2y^3 = y \frac{dx}{dy} - x$

dividing y on both sides

$\frac{2y^{\cancel{3}}}{\cancel{y}} = \frac{y}{y} \frac{dx}{dy} - \frac{x}{y}$

$2y^2 = \frac{dx}{dy} - \frac{x}{y}$

$$\frac{dx}{dy} - x\frac{1}{y} = 2y^2$$

$$P = -\frac{1}{y} \quad Q = 2y^2$$

$$IF = e^{\int P dy}$$

$$= e^{\int -\frac{1}{y} dy}$$

$$= e^{-\int \frac{1}{y} dy}$$

$$= e^{-(\log e^{y})}$$

$$= y^{-1}$$

$$IF = \frac{1}{y}$$

General solution $= x(IF) = \int Q(y)\,(IF)\, dy + c$

$$\cancel{x\left(\frac{1}{y}\right) = \int 2y^2 \left(\frac{1}{y}\right) dy + c}$$

$$x\left(\frac{1}{y}\right) = \int 2y^{\cancel{2}}\left(\frac{1}{\cancel{y}}\right) dy + c$$

$$= \int 2y \, dy + c$$

$$= 2\int y \, dy + c$$

$$= \cancel{2}\frac{y^2}{\cancel{2}} + c$$

$$\frac{x}{y} = y^2 + c$$

4 Solve $(1+y^2)\,dx = (\tan^{-1}y - x)\,dy$

$$(1+y^2)\frac{dx}{dy} = \tan^{-1}y - x$$

$$(1+y^2)\frac{dx}{dy} + x = \tan^{-1}y$$

dividing $(1+y^2)$ on both sides

$$\frac{dx}{dy} + x\frac{1}{1+y^2} = \frac{\tan^{-1}y}{1+y^2}$$

$$P = \frac{1}{1+y^2}, \quad Q = \frac{\tan^{-1}y}{1+y^2}$$

$$IF = e^{\int P dy}$$

$$= e^{\int \frac{1}{1+y^2} dy}$$

$$= e^{\tan^{-1}y}$$

General solution $x(IF) = \int Q(y)(IF)\,dy + c$

$$x\left(e^{\tan^{-1}y}\right) = \int \frac{\tan^{-1}y}{1+y^2}\left(e^{\tan^{-1}y}\right)dy + c$$

$$= \int \tan^{-1}y \, \frac{1}{1+y^2}\left(e^{\tan^{-1}y}\right)dy + c$$

$$\boxed{\begin{aligned} &\tan^{-1}y = t \\ &\frac{1}{1+y^2}\,dy = dt \\ &dy = dt\,(1+y^2) \end{aligned}}$$

$$= \int dt\, t\, e^t + c$$

$$= \int t\, e^t\, dt + c$$

$$= t\int e^t dt - \int\left[\frac{d}{dt}t \int e^t dt\right] dt + c$$

$$= t e^t - \int 1\, e^t\, dt + c$$

$$= t e^t - e^t + c$$

$$x\, e^{\tan^{-1}y} = \tan^{-1}y\, e^{\tan^{-1}y} - e^{\tan^{-1}y} + c$$

Change of variables

If the given differential equation does not directly come under any of the forms discussed so far to one of these forms, then we can reduce the differential equation by suitable substitution.

This procedure of reducing the given differential equation by substitution is called the change of (dependent or independent) variables.

Solve $\frac{dy}{dx} + \frac{y}{x} \log y = \frac{y}{x^2} (\log y)^2$

Dividing both sides of the given equation by $y(\log y)^2$ we have

$$\frac{1}{y(\log y)^2} \frac{dy}{dx} + \frac{1}{x} \cdot \frac{1}{\log y} = \frac{y}{x^2} \frac{\log (y^2)}{y(\log y^2)}$$

$$\frac{1}{y(\log y)^2} \frac{dy}{dx} + \frac{1}{x} \frac{1}{\log y} = \frac{1}{x^2} \longrightarrow (1)$$

Now put $\frac{1}{\log y} = u$, so that

$$\frac{du}{dx} = -\frac{1}{(\log y)^2} \cdot \frac{1}{y} \cdot \frac{dy}{dx}$$

$$-\frac{du}{dx} = \frac{1}{(\log y)^2} \cdot \frac{1}{y} \frac{dy}{dx}$$

Substitutions the equation (1)

$$-\frac{du}{dx} + \frac{1}{x} u = \frac{1}{x^2} \Rightarrow \frac{du}{dx} - \frac{1}{x} u = -\frac{1}{x^2}$$

Here $P = -\frac{1}{x}$ and $Q = -\frac{1}{x^2}$

$$\therefore IF = e^{\int P dx}$$

$$= e^{-\int \frac{1}{x} dx} \Rightarrow e^{-(\log x)} \Rightarrow e^{\log x^{-1}}$$

$$= x^{-1}$$

$$IF = \frac{1}{x}$$

General solution is $u(IF) = \int Q(IF)\, dx + c$

$$u \frac{1}{x} = \int -\frac{1}{x^2} \left(\frac{1}{x}\right) dx + c$$

$$\frac{u}{x} = -\int \frac{1}{x^3} dx + c$$

$$\frac{1}{\log y} \cdot \frac{1}{x} = -\int x^{-3} dx + c$$

$$\frac{1}{\log y}\cdot\frac{1}{x} = -\left[\frac{x^{-3+1}}{-3+1}\right]+c$$

$$\frac{1}{\log y}\cdot\frac{1}{x} = -\left[\frac{x^{-2}}{-2}\right]+c$$

$$\frac{1}{\log y}\cdot\frac{1}{x} = -\left[-\frac{1}{2x^2}\right]+c$$

$$\frac{1}{\log y}\cdot\frac{1}{x} = \frac{1}{2x^2}+c$$

Solve $\frac{dy}{dx}+\frac{1}{x}=\frac{e^y}{x^2}$

on dividing out by e^y, the given equation becomes e^{-y}

$$e^{-y}\frac{dy}{dx}+e^{-y}\frac{1}{x}=\frac{1}{x^2} \longrightarrow (1)$$

Now put $e^{-y}=u$ so that $\frac{d}{dx}e^{-y}=-1e^{-y}\frac{dy}{dx}$

$$\frac{du}{dx}=-e^{-y}\frac{dy}{dx} \qquad \therefore \frac{d}{dx}(e^{ax})=ae^{ax}\frac{dx}{dy}$$

$$-\frac{du}{dx}=e^{-y}\frac{dy}{dx}$$

Substitute in these eq (1)

$$-\frac{du}{dx}+\frac{1}{x}u=\frac{1}{x^2}$$

dividing by -1

$$\frac{du}{dx}-\frac{1}{x}u=-\frac{1}{x^2}$$

Here $P=-\frac{1}{x}$, $Q=-\frac{1}{x^2}$

$$\therefore IF = e^{\int P dx} = e^{-\int\frac{1}{x}dx} = e^{-\log x} = e^{\log\frac{1}{x}}$$

$$IF=\frac{1}{x}$$

G.S $v(IF) = \int (Q(IF))\, dx + c$

$$v\frac{1}{x} = -\int \frac{1}{x^2}\cdot\frac{1}{x}\, dx + c$$

$$= -\int x^{-2} x^{-1}\, dx + c$$

$$= -\left[\frac{x^{-3+1}}{-3+1}\right] + c$$

$$= -\left[\frac{x^{-2}}{-2}\right] + c$$

$$= -\left[-\frac{1}{2x^2} + c\right]$$

$$e^{-y}\frac{1}{x} = \frac{1}{2x^2} + c$$

$$\frac{1}{e^{y}}\frac{1}{x} = \frac{1+2cx^2}{2x^2}$$

$$\frac{2x^{\cancel{2}}}{\cancel{x}} = (1+2cx^2)\, e^{y}$$

$$2x = (1+2cx^2)\, e^{y}$$

Solve $(x+1)\frac{dy}{dx} + 1 = e^{x-y}$

Given equation is $(x+1)\frac{dy}{dx} + 1 = e^{x-y}$

$$(x+1)\frac{dy}{dx} + 1 = e^{x}\cdot e^{-y}$$

$$\frac{x+1}{e^{-y}}\frac{dy}{dx} + \frac{1}{e^{-y}} = e^{x}$$

$x+1$ ~~diving~~ dividing on both sides

$$\frac{1}{e^{-y}}\frac{dy}{dx} + \frac{1}{e^{-y}}\frac{1}{x+1} = \frac{e^{x}}{x+1}$$

$$e^{y}\frac{dy}{dx} + \frac{1}{x+1}e^{y} = \frac{e^{x}}{x+1} \longrightarrow (1)$$

$$v = e^{y}$$

$$\frac{dv}{dx} = e^{y}\frac{dy}{dx}$$

Substitute eq (1)

$$\frac{du}{dx} + \frac{1}{x+1} u = \frac{e^x}{x+1}$$

comparing with

$$\frac{du}{dx} + P(x) u = Q(x)$$

$$P = \frac{1}{x+1} \quad Q = \frac{e^x}{x+1}$$

$$IF = e^{\int P dx}$$

$$= e^{\int \frac{1}{x+1} dx} = e^{\log(1+x)} = 1+x$$

G.S $u(IF) = \int Q(IF)\, dx + c$

$$u(1+x) = \int \frac{e^x}{\cancel{x+1}} \cancel{(1+x)}\, dx + c$$

$$e^y(1+x) = e^x + c$$

Solve $\frac{dy}{dx} + (2x\,\tan^{-1}y - x^3)(1+y^2) = 0$

$$\frac{dy}{dx} = (-2x\,\tan^{-1}y + x^3)(1+y^2)$$

dividing by $(1+y^2)$ on both sides

$$\left(\frac{1}{1+y^2}\right)\frac{dy}{dx} = \frac{(-2x\,\tan^{-1}y + x^3)(1+y^2)}{1+y^2}$$

$$\frac{1}{1+y^2}\frac{dy}{dx} = -2x\,\tan^{-1}y + x^3$$

$$\frac{1}{1+y^2}\frac{dy}{dx} + 2x\,\tan^{-1}y = x^3 \longrightarrow (1)$$

put $u = \tan^{-1}y$

$$\frac{du}{dx} = \frac{1}{1+y^2}\frac{dy}{dx}$$

substitude $\frac{du}{dx}$ in eq (1)

$$\frac{du}{dx} + 2x\,u = x^3$$

comparing with $\frac{du}{dx} + P(x)y = Q(x)$

$P = 2x \quad Q = x^3$

$IF = e^{\int P dx} = e^{\int 2x\, dx}$

$= e^{2\int x dx} = e^{2\left(\frac{x^2}{2}\right)} = e^{x^2}$

General Solution $u(IF) = \int Q(IF)\, dx + c$

$u e^{x^2} = \int x^3 (e^{x^2})\, dx + c$

$\boxed{\begin{array}{l} x^2 = t \\ dt = 2x\, dx \\ \frac{dt}{2x} = dx \end{array}}$

$\tan^{-1} y\, e^{x^2} = \int t\, e^t \frac{dt}{dx} + c$

$= \frac{1}{2} \int \underset{u}{t}\, \underset{v}{e^t}\, dt + c$

$= \frac{1}{2} \left[\overset{u}{t} \int \overset{v}{e^t} dt - \left[\int \frac{\overset{u}{dt}}{dt} \int \overset{v}{e^t} dt \right] \right] + c$

$= \frac{1}{2} \left(t e^t - \int e^t dt \right) + c$

$\tan^{-1} y\, e^{x^2} = \frac{1}{2} \left(t e^t - e^t \right) + c$

$\tan^{-1} y\, e^{x^2} = \frac{x^2 e^{x^2} - e^{x^2}}{2} + c$

$2 \tan^{-1} y\, e^{x^2} = e^{x^2} (x^2 - 1) + 2c$

Solve $\frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$

$\frac{1}{\cos^2 y} \frac{dy}{dx} + x \frac{\sin 2y}{\cos^2 y} = x^3$

$\sec^2 y \frac{dy}{dx} + x \frac{2 \sin y \cancel{\cos y}}{\cos^{\cancel{2}} y} = x^3$

$\sec^2 y \frac{dy}{dx} + 2x \tan y = x^3 \longrightarrow (1)$

$\tan y = u$

$\frac{du}{dx} = \sec^2 y \frac{dy}{dx}$

Substitute in this eq (1)

$\frac{du}{dx} + 2x\,u = x^3$

Comparing with $\frac{du}{dx} + P\,u = Q$

$P = 2x \quad Q = x^3$

$IF = e^{\int P\,dx}$

$= e^{\int 2x\,dx} = e^{2\int x\,dx} = e^{\cancel{2}\left(\frac{x^2}{\cancel{2}}\right)} = e^{x^2}$

General solution $u\,IF = \int Q\,(IF)\,dx + c$

$u\,e^{x^2} = \int x^3 (e^{x^2})\,dx + c$

$x^2 = t$

$2x\,dx = dt$

$dx = \frac{dt}{2x}$

$= \int x^{\cancel{3}\,2} e^t \frac{dt}{2\cancel{x}} + c$

$= \int \frac{t e^t\,dt}{2} + c$

$\int u.v\,dx = u\int v\,dx - \left[\int \frac{d}{dx} u \int v\,dx\right] + c$

$= \frac{1}{2}\int t\,e^t\,dt + c$

$= \frac{1}{2}\left[t \int e^t dt - \left[\int \frac{d}{dt} t \int e^t dt\right]\right] + c$

$= \frac{1}{2}\left[t e^t - \int e^t dt\right] + c$

$= \frac{1}{2}\left(t e^t - e^t\right) + c$

$= \frac{1}{2}\left(x^2 e^{x^2} - e^{x^2}\right) + c$

$\tan y\, e^{x^2} = \frac{e^{x^2}(x^2-1)}{2} + c$

$\tan y = \frac{\cancel{e^{x^2}}(x^2-1)}{2\,\cancel{e^{x^2}}} + \frac{c}{e^{x^2}}$

$\tan y = \frac{x^2}{2} - \frac{1}{2} + c e^{-x^2}$

Bernoulli's equation

An equation of the form $\frac{dy}{dx} + Py = Qy^n$, where P and Q are functions of x only is called a Bernoulli's equation

$$\frac{dy}{dx} + P(x)y = Q(x)\, y^n$$

dividing by y^n

$$\frac{1}{y^n}\frac{dy}{dx} + P(x)\frac{y}{y^n} = Q(x)$$

$$\frac{1}{y^n}\frac{dy}{dx} + P(x)\frac{1}{y^{n-1}} = Q(x)$$

$$U = \frac{1}{y^{n-1}} \Rightarrow y^{-n+1}$$

$$\frac{du}{dx} = (-n+1)\, y^{-n+1-1}\,\frac{dy}{dx}$$

$$\frac{du}{dx} = (-n+1)\, y^{-n}\,\frac{dy}{dx}$$

$$= (1-n)\frac{1}{y^n}\frac{dy}{dx}$$

$$\frac{1}{1-n}\frac{du}{dx} = \frac{1}{y^n}\frac{dy}{dx} \Rightarrow \frac{1}{y^n}\frac{dy}{dx} + P(x)\frac{1}{y^{n-1}} = Q(x)$$

$$\frac{1}{1-n}\frac{du}{dx} + P(x)\, U = Q(x)$$

$$\frac{du}{dx} + (1-n)\, P(x)\, U = (1-n)\, Q(x)$$

$$\frac{du}{dx} + P_1(x)\, U = Q_1(x)$$

$$P_1(x) = (1-n)\, P(x)$$

$$Q_1(x) = (1-n)\, Q(x)$$

Solve $\frac{dy}{dx} + \frac{y}{x} = x^2 y^6$

Dividing by y^6

$\frac{1}{y^6} \frac{dy}{dx} + \frac{y}{x} \cdot \frac{1}{y^6} = x^2$

$\frac{1}{y^6} \frac{dy}{dx} + \frac{1}{x} \cdot \frac{1}{y^5} = x^2$

$u = \frac{1}{y^5} = y^{-5}$

$\frac{du}{dx} = -5 y^{-5-1} \frac{dy}{dx}$

$\frac{du}{dx} = -5 y^{-6} \frac{dy}{dx} \Rightarrow -5 \frac{1}{y^6} \frac{dy}{dx}$ dividing -5

$\frac{1}{-5} \frac{du}{dx} = \frac{1}{y^6} \frac{dy}{dx}$

$-\frac{1}{5} \frac{du}{dx} + \frac{1}{x} u = x^2$

multiplicate (-5) on both sides

$\frac{du}{dx} - \frac{5}{x} u = -5x^2$

$P = \frac{-5}{x} \quad u = -5x^2$

$IF = e^{\int P dx} = e^{\int -\frac{5}{x} dx} = e^{-5 \int \frac{1}{x} dx}$

$= e^{-5(\log e^x)} = e^{\log e^{x^{-5}}}$

$= x^{-5} = \frac{1}{x^5}$

General solution $= u(IF) = \int Q (IF) dx + c$

$u \left(\frac{1}{x^5} \right) = \int -5x^2 \left(\frac{1}{x^5} \right) dx + c$

$u \left(\frac{1}{x^5} \right) = -5 \int x^2 \frac{1}{x^5} dx + c$

$$u\left(\frac{1}{x^5}\right) = -5\int \frac{1}{x^3}dx + c$$
$$= -5\int x^{-3}dx + c$$
$$= -5\left(\frac{x^{-3+1}}{-3+1}\right) + c$$
$$= -5\left(\frac{x^{-2}}{-2}\right) + c$$

$$u\left(\frac{1}{x^5}\right) = \frac{5}{2}\frac{1}{x^2} + c$$
$$u = \frac{5}{2}\frac{x^5}{x^2} + cx^5$$
$$\frac{1}{y^5} = \frac{5}{2}x^3 + cx^5$$

5 Solve $x\frac{dy}{dx} + y = y^2 \log x$

dividing by y^2

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{y}{x}\frac{1}{y^2} = \frac{\log x}{x}$$

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{x}\frac{1}{y} = \frac{\log x}{x} \longrightarrow (1)$$

$$u = \frac{1}{y} = y^{-1}$$

$$\frac{du}{dx} = -1y^{-1-1}\frac{dy}{dx}$$

$$\frac{du}{dx} = -1y^{-2}\frac{dy}{dx}$$

$$\frac{du}{dx} = -\frac{1}{y^2}\frac{dy}{dx}$$

$$-\frac{du}{dx} = \frac{1}{y^2}\frac{dy}{dx}$$

substitute in these value in eq (1)

$$-\frac{du}{dx} + \frac{1}{x}u = \frac{\log x}{x}$$

dividing by x

$$x\frac{dy}{dx} + \frac{y}{x} = \frac{y^2\log x}{x}$$ dividing by y^2

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{y}{x}\frac{1}{y^2} = \frac{\log x}{x}$$

$$u = \frac{1}{y} = y^{-1} \quad u = y^{-1}$$

$$\frac{du}{dx} = -1y^{-1-1}\frac{dy}{dx}$$

multiple -1 on both sides

$\frac{du}{dx} - \frac{1}{x}u = -\frac{\log x}{x} \Rightarrow P = -\frac{1}{x} \quad Q = -\frac{\log x}{x}$

$IF = e^{\int P dx} = e^{-\int \frac{1}{x} dx} = e^{-(\log_e x)}$

$= e^{\log_e x^{-1}}$

$IF = x^{-1} = \frac{1}{x}$

G.S $\quad u(IF) = \int Q(IF)\, dx + c$

$u\left(\frac{1}{x}\right) = \int -\frac{\log x}{x}\left(\frac{1}{x}\right) dx + c$

$\frac{u}{x} = -\int \log x \frac{1}{x^2} dx + c$

$= -\left[\log x \int \frac{1}{x^2} dx - \int \frac{d}{dx} \log x \int \frac{1}{x^2} dx\right] + c$

$= -\left[\log x\left(\frac{x^{-2+1}}{-2+1}\right) - \int \frac{1}{x}\left[\frac{x^{-2+1}}{-2+1}\right] dx + c\right]$

$= -\left[\log x \frac{x^{-1}}{-1} - \int \frac{1}{x} \frac{x^{-1}}{-1} dx + c\right]$

$= -\left[\log x \frac{x^{-1}}{-1} - \int \frac{1}{x} - \left(\frac{1}{x}\right) dx + c\right]$

$= -\left[\log x \frac{x^{-1}}{-1} - \int -\frac{1}{x^2} dx + c\right]$

$= -\left[\log x\left(-\frac{1}{x}\right) - \left(-\frac{x^{-2+1}}{-2+1}\right)\right]$

$= -\left(-\frac{1}{x}\log x + \frac{x^{-1}}{-1} + c\right)$

$= -\left[-\frac{1}{x}\log x - \frac{1}{x} + c\right]$

$= \frac{1}{x}\log x + \frac{1}{x} + c$

$= \frac{1}{x}\left[\log x + 1\right] + c$

Solve $= 2y \tan x + y^2 \tan^2 x$

$\frac{dy}{dx} - 2y \tan x = y^2 \tan^2 x$

dividing y^2 on both sides

$\frac{1}{y^2}\frac{dy}{dx} - (2\tan x)\frac{y}{y^2} = \frac{y^2}{y^2}\tan^2 x$

$\frac{1}{y^2}\frac{dy}{dx} - 2\tan x \frac{1}{y} = \tan^2 x$

$u = \frac{1}{y} = y^{-1}$

$$\frac{du}{dx} = -1\, y^{-1-1} \frac{dy}{dx}$$

$$\frac{du}{dx} = -1\, y^{-2} \frac{dy}{dx}$$

$$-\frac{du}{dx} = \frac{1}{y^2} \frac{dy}{dx}$$

$$-\frac{du}{dx} - \tan x\, u = \tan^2 x$$

multiple - on both sides

$$\frac{du}{dx} + 2\tan x\, u = -\tan^2 x$$

$$\int \tan x\, dx = \int \frac{\sin x}{\cos x} dx$$

$$t = \cos x$$

$$dt = -\sin x\, dx$$

$$dx = \frac{dt}{-\sin x}$$

$$= \int \frac{\sin x}{t} \frac{dt}{-\sin x}$$

$$= -\int \frac{dt}{t}$$

$$= -\log|t|$$

$$= \log\left(\frac{1}{t}\right)$$

$$= \log\left(\frac{1}{\cos x}\right)$$

$$= \log|\sec x|$$

comparing with $\frac{du}{dx} + P(x)\, y = Q(x)\, y^n$

$$P = 2\tan x \qquad Q = -\tan^2 x$$

$$IF = e^{\int P dx} = e^{\int 2 \tan x\, dx} = e^{2\int \tan x\, dx}$$

$$= e^{2(\log|\sec x|)}$$

$$e^{\int P dx} = 2\sec^2 x$$

General solution $u(IF) = \int Q\,(IF)\, dx + C$

$$u \sec^2 x = \int -\tan^2 x\, (\sec^2 x)\, dx + C$$

$$\therefore \tan x = t$$

$$\sec^2 x\, dx = dt$$

$$= \int -t^2\, dt + C$$

$$= \left[-\frac{t^{2+1}}{2+1}\right] + C$$

$$u \sec^2 x = -\frac{t^3}{3} + C$$

$$\frac{1}{y} \sec^2 x = -\frac{\tan^3 x}{3} + C$$

$$\frac{\sec^2 x}{y} = -\frac{\tan^3 x}{3} + C$$

Solve $\frac{dy}{dx} + \frac{y}{x} = y^2 x \sin x$

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{x}\frac{y}{y^2} = \frac{y^2}{y^2} x \sin x$$

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{x}\frac{1}{y} = x \sin x$$

$$u = \frac{1}{y} = y^{-1}$$

$$\frac{du}{dx} = -1\, y^{-1-1} \frac{dy}{dx}$$

$$\frac{du}{dx} = -1\, y^{-2} \frac{dy}{dx}$$

$$-\frac{du}{dx} = \frac{1}{y^2}\frac{dy}{dx}$$

$$-\frac{du}{dx} + \frac{1}{x} u = x \sin x$$

$$\frac{du}{dx} - \frac{1}{x} u = -x \sin x$$

comparing with $\frac{du}{dx} + P(x) y = Q(x) y$

$$P = -1/x \quad Q = -x \sin x$$

$$IF = e^{\int P dx} = e^{\int -\frac{1}{x} dx} = e^{-\log e^{x}} = e^{\log e^{x^{-1}}}$$

$$= x^{-1} = \frac{1}{x}$$

General solution $u(IF) = \int Q\,(IF)\, dx + c$

$$u\frac{1}{x} = \int -x \sin x \left(\frac{1}{x}\right) dx + c$$

$$u\frac{1}{x} = -\int \frac{\cancel{x} \sin x}{\cancel{x}} dx + c$$

$$u\frac{1}{x} = -\int \sin x \, dx + c$$

$$\frac{1}{y}\frac{1}{x} = -[-\cos x] + c$$

$$\frac{1}{xy} = \cos x + c$$

$$1 = xy \cos x + cxy \Rightarrow xy \cos x + cxy - 1 = 0$$

Exact Differential Equations

A differential equation $Mdx + Ndy = 0$ where M, N are functions of x, y is said to be exact if there exists a function u of x, y such that $Mdx + Ndy = dU$.

Theorem:- The necessary and sufficient condition for the differential equation $Mdx + Ndy = 0$ to be exact is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$$Mdx + Ndy = 0 \quad -①$$

If $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Then equation ① is exact

compute $V = \int_{y\,\text{const}} M\,dx$, $\frac{\partial V}{\partial y}$

compute $B = \int \left(N - \frac{\partial V}{\partial y}\right) dy$

General solution as $V + B = c$

Solve $(4x + 3y + 1)\,dx + (3x + 2y + 1)\,dy = 0$

$M = 4x + 3y + 1$

$N = 3x + 2y + 1$

$$\frac{\partial M}{\partial y} = 4(0) + 3(1) + 0$$

$$\frac{\partial M}{\partial y} = 3$$

$$\frac{\partial N}{\partial x} = 3(1) + 2(0) + 0$$

$$\frac{\partial N}{\partial x} = 3$$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, Given equation is exact

$V = \int M\,dx$
y is constant

$= \int (4x+3y+1)\,dx$

$= \int 4x\,dx + \int (3y+1)\,dx$

$= 4\int x\,dx + (3y+1)\int dx$

$= \cancel{4}^{2}\frac{x^2}{\cancel{2}} + (3y+1)x \Rightarrow 2x^2+3xy+x$

$\frac{\partial V}{\partial y} = 0 + 3x \cdot 1 + 0 = 3x$

$B = \int (3x+2y+1-3x)\,dy$

$= \int (2y+1)\,dy$

$= \int 2y\,dy + \int dy$

$= 2\int y\,dy + y$

$= 2\frac{y^2}{2} + y$

$= y^2 + y$

$V + B = C$

$2x^2 + 3xy + x + y^2 + y = C$

Solve $(hx+by+f)dy + (ax+hy+g)dx = 0$

$M = ax + hy + g$

$N = hx + by + f$

$\frac{\partial M}{\partial y} = a(0) + h(1) + 0$

$\frac{\partial M}{\partial y} = h$

$\frac{\partial N}{\partial x} = h(1) + b(0) + 0$

$\frac{\partial N}{\partial x} = h$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ given equation is exact

$v = \int M dx$

$v = \int_{y\,constant} (ax + hy + g)\, dx$

$= \int ax\, dx + \int (hy+g)\, dx$

$= a\int x\, dx + (hy+g)\int dx$

$= a\frac{x^2}{2} + (hy+g)\, x$

$v = \frac{ax^2}{2} + hxy + gx$

$\frac{\partial v}{\partial y} = \frac{a(0)}{2} + hx(1) + g(0)$

$= hx$

$B = \int \left(N - \frac{\partial v}{\partial y}\right) dy$

$= \int (\cancel{hx} + by + f - \cancel{hx})\, dy$

$= \int (by + f)\, dy$

$= \int by\, dy + \int f\, dy$

$= b\int y\, dy + f\int dy$

$= b\frac{y^2}{2} + fy$

$v + B = c$

$\frac{ax^2}{2} + hxy + gx + \frac{by^2}{2} + fy = c$

Solve $(x^2 - 2xy - y^2)\, dx - (x+y)^2\, dy = 0$

$M = x^2 - 2xy - y^2$

$N = -(x^2 + 2xy + y^2)$

$\frac{\partial M}{\partial y} = 0 - 2(1) - (1)^2$

$= -2 + 1 = -3$

$\frac{\partial N}{\partial x} = -(1^2 + 2(1) + 0) = -1 - 2 = -3$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ given equation is exact

$$v = \int_{y\text{-con}} M\,dx$$

$$= \int_{y\text{-con}} x^2 - 2xy - y^2$$

$$= \int x^2\,dx - \int (2xy + y^2)\,dx$$

$$= \frac{x^3}{3} - \int (2xy + y^2) \int dx$$

$$= \frac{x^3}{3} - (2x^2y + xy^2)$$

$$= \frac{x^3}{3} - 2x^2y - xy^2$$

Solve $(x^2 - 2xy - y^2)\,dx - (x+y)^2\,dy = 0$

$(x^2 - 2xy - y^2)\,dx - (x + 2xy + y^2)\,dy = 0$

$$M = x^2 - 2xy - y^2$$

$$N = -(x^2 + 2xy + y^2)$$

$$\frac{\partial M}{\partial y} = 0 - 2x(1) - 2y$$

$$= -2x - 2y$$

$$\frac{\partial N}{\partial x} = -2x - 2y - 0$$

$$= -2x - 2y$$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ Given equation is exact

$$v = \int_{y\text{-con}} M\,dx$$

$$= \int (x^2 - 2xy - y^2)\,dx$$

$$= \int x^2\,dx - \int 2xy\,dx - \int y^2\,dx$$

$$= \frac{x^3}{3} - 2y \int x\,dx - y^2 \int dx$$

$$= \frac{x^3}{3} - \cancel{2}y\,\frac{x^2}{\cancel{2}} - y^2x$$

$= \frac{x^3}{3} - yx^2 - y^2x$

$\frac{\partial v}{\partial y} = 0 - 1 \cdot x^2 - 2yx$

$= -x^2 - 2yx$

$B = \int (N - \frac{\partial v}{\partial y}) dy$

$= \int (-x^2 - 2xy - y^2 - [-x^2 - 2xy]) dy$

$= \int (-x^2 - 2xy - y^2 + x^2 + 2xy) dy$

$= \int -y^2 dy$

$= -\int y^2 dy$

$= -\frac{y^3}{3}$

$x^2 + K_1x + K_2$

$\frac{d}{dx} = 2x + K_1 + 0$

$v + B = c$

$\frac{x^3}{3} - yx^2 - y^2x - \frac{y^3}{3} = c$

(1) Solve $(1+e^{x/y})dx + (1-\frac{x}{y}) e^{x/y} dy = 0$

$M = 1 + e^{x/y}$

$N = (1 - x/y) e^{x/y}$

$\frac{\partial M}{\partial y} = 0 + e^{x/y} \cdot x \left(-\frac{1}{y^2}\right)$

$= -\frac{x}{y^2} e^{x/y} \longrightarrow (1)$

$\frac{\partial N}{\partial x} = (1 - x/y) \frac{\partial}{\partial x} e^{x/y} + e^{x/y} \frac{\partial}{\partial x} (1 - x/y)$

$= (1 - x/y) e^{x/y} (\frac{1}{y}) + e^{x/y} (0 - 1/y)$

$= 1 \cdot e^{x/y} \frac{1}{y} - \frac{x}{y} e^{x/y} (1/y) - \frac{1}{y} e^{x/y}$

$= e^{x/y} \frac{1}{y} - \frac{x}{y^2} e^{x/y} - e^{x/y} \frac{1}{y}$

$= -\frac{x}{y^2} e^{x/y} \longrightarrow (2)$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Given equation is exact

$$v = \int_{y\text{-con}} M\,dx$$

$$v = \int 1 + e^{x/y}\,dx$$

$$= \int dx + \int e^{x/y}\,dx$$

$$= x + \frac{e^{x/y}}{\frac{1}{y}} \Rightarrow x + y e^{x/y}$$

$$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\left(x + y e^{x/y}\right)$$

$$= 0 + \frac{\partial}{\partial y}\left(y e^{x/y}\right)$$

$$= y \frac{\partial}{\partial y}\left(e^{x/y}\right) + e^{x/y} \frac{\partial}{\partial y}(y)$$

$$= y e^{x/y}\left(-\frac{x}{y^2}\right) + e^{x/y}$$

$$= -\frac{x}{y} e^{x/y} + e^{x/y}$$

$$= \left(1 - \frac{x}{y}\right) e^{x/y}$$

$$B = \int \left(N - \frac{\partial v}{\partial y}\right) dy$$

$$= \int \left[\left(1 - x/y\right) e^{x/y} - \left(1 - x/y\right) e^{x/y}\right] dy$$

$$= \int 0\,dy = c$$

$$v + B = c \Rightarrow x + y e^{x/y} = c$$